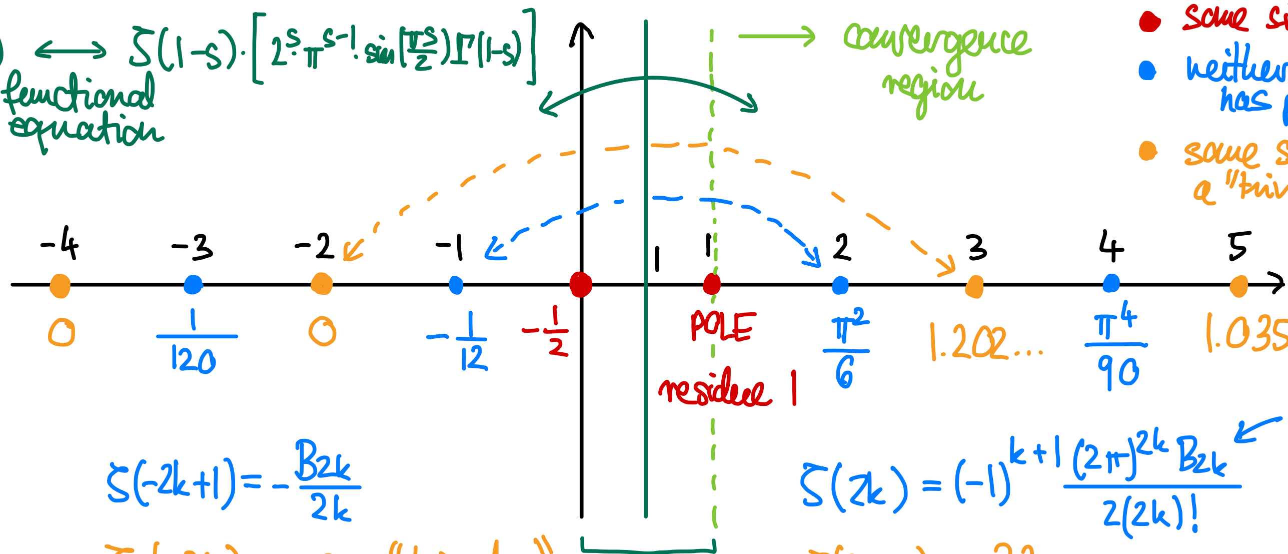
SPECIAL VALUES OF L-FUNCTIONS

Example 1. Riemann ζ-function: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, for $Re(s) > 1$

$\zeta(s) \longleftrightarrow \zeta(1-s)\cdot\left[2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\right]$

functional equation

● some side has pole
● neither side has pole or zero
● some side has a "trivial zero"



$\zeta(-2k+1) = -\frac{B_{2k}}{2k}$

$\zeta(-2k) = 0$ "trivial 0"

$\zeta'(-2k) = ??$

ANALYTIC NUMBER THEORY

$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}$ ← Bernoulli numbers

$\zeta(2k+1) = ??$

E.g. $\zeta(3) = \sum \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$

Apery: $\zeta(3) \notin \mathbb{Q}$

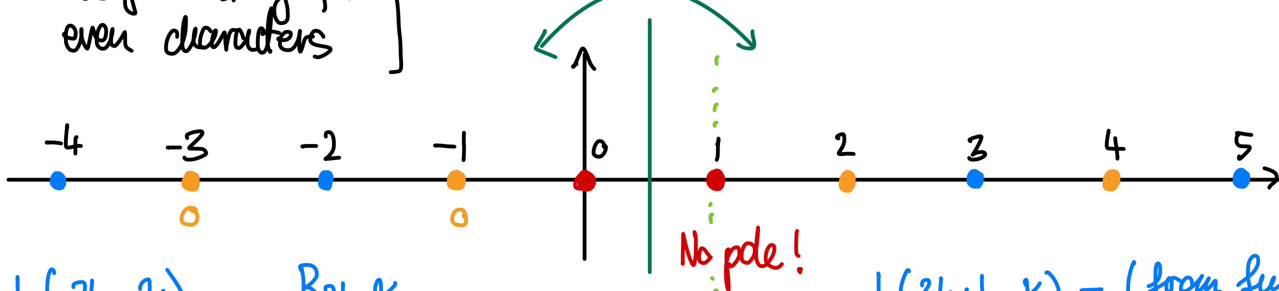
Example 2. Dirichlet L-functions.

$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times \rightsquigarrow L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ for $Re(s) > 1$

odd Dirichlet character $\chi(-1) = -1$

[∃ analogous story for even characters]

$L(1-s,\chi) \sim L(s,\chi)$



$L(-2k,\chi) = -\frac{B_{2k,\chi}}{2k}$

twisted Bernoulli numbers

$L(2k+1,\chi) = $ (from functional equation)

CLASS NUMBER FORMULA

Dirichlet (1839): For $d < 0$,

$L\left(0, \left(\frac{\cdot}{d}\right)\right) = h_{\mathbb{Q}(\sqrt{-d})}$.

$L(-2k+1,\chi) = 0$

$L'(-2k+1,\chi) = ??$

$L(2k,\chi) = ??$ (CATALAN CONSTANT)

E.g. $L\left(2, \left(\frac{\cdot}{-4}\right)\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$

Example 3. Dedekind ζ-functions.

F = number field $\rightsquigarrow$ $\zeta_F(s) = \sum_{\mathfrak{a}} \left| \mathcal{O}_F/\mathfrak{a} \right|^{-s}$

Thm (Class Number Formula). Dedekind/Landau (1903), Hecke (1917)

$$\underbrace{\zeta_F^*(0)}_{\text{Leading term at } 0} = -\frac{h_F \cdot R_F}{w_F}$$

h_F = class group

R_F = regulator of $\mathcal{O}_F^\times$

w_F = # of roots of unity in F

A huge generalization was found by Borel:

Thm (Borel, 1970s). $\forall n > 1$, $\exists\, q_n \in \mathbb{Q}^\times$ s.t.

$$\zeta_F^*(1-n) = q_n \cdot R_n, \qquad R_n = \text{regulator of } K_{2n-1}(\mathcal{O}_F)$$

("K-theory of $\mathcal{O}_F$")

(Note: sometimes $R_n = 1 \Rightarrow$ • instead of •.)

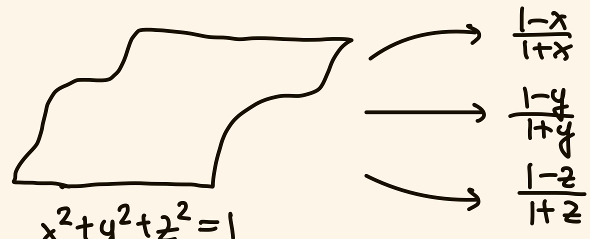
→ Beilinson (~1985): general conjecture L-values ~ K-theory

→ Bloch (~1986): rephrasing in terms of "higher Chow groups"
(also 1979...)

Thm. (H.–A'Campo).

(1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} = L\left(2, \left(\frac{\cdot}{-4}\right)\right)$

is explicitly related to:



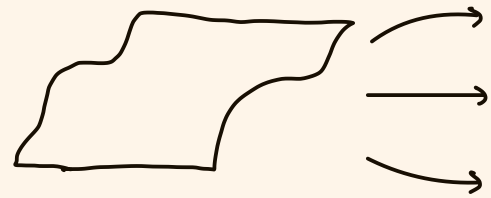
$\rightsquigarrow$ higher Chow group of $x^2+y^2+z^2=1$

Explicitly:

$$\int_{X(\mathbb{R})} \log\left(\frac{1-x}{1+x}\right) d\log\left(\frac{1-y}{1+y}\right) d\log\left(\frac{1-z}{1+z}\right)$$

(2) $\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3)$

is explicitly related to:



$\frac{1-x_1}{1+x_1}$

$\frac{1-x_2}{1+x_2}$

$\vdots$

$x_1^2 + \cdots + x_5^2 = 1$

A different direction of generalizing the Class Number Formula was investigated by Birch & Swinnerton-Dyer.

Example 4. Elliptic curves.

$a, b \in \mathbb{Q}$

$E : y^2 = x^3 + ax + b$

$\Delta = 4a^3 + 27b^2 \neq 0$

$E(\mathbb{Q}) = \{(x,y) \in \mathbb{Q}^2 : y^2 = x^3 + ax + b\} \cup \{O\}$

$\cong \mathbb{Z}^r \oplus (\text{torsion})$ (Mordell-Weil Theorem)

$E(\mathbb{F}_p) = \{(x,y) \in \mathbb{F}_p^2 : y^2 \equiv x^3 + ax + b \mod p\} \cup \{O\}$

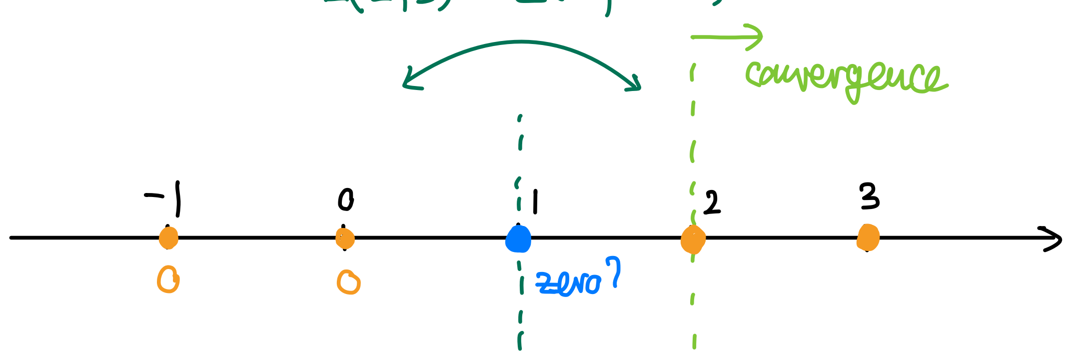
$|E(\mathbb{F}_p)| - (p+1) =: \underbrace{a_p(E)}_{\text{error}}$ $\qquad |a_p(E)| \leq 2\sqrt{p}$

Building on ideas of Weil, define for $\mathrm{Re}(s) > 2$:

$$L(E, s) = \prod_{p \nmid \Delta} \left(1 - a_p(E)\, p^{-s} + p^{1-2s}\right)^{-1} = \sum_{n=1}^{\infty} a_n(E)\, n^{-s}.$$

(Wiles/Taylor-Wiles gives...)

$L(E,s) \sim L(E, 2-s)$



Originally: • ⇝ • (Class Number Formula), but actually it's •

BSD Conj.

- $\operatorname{ord}_{s=1} L(E,s) = \text{rank } r$ from $E(\mathbb{Q}) = \mathbb{Z}^r \oplus (\text{torsion})$

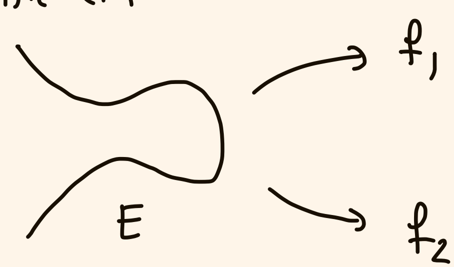
  More analytically:
  $$s=1 \rightsquigarrow \left(1 - a_p(E)p^{-1} + p^{1-2}\right) = \frac{p+1-a_p(E)}{p} = \frac{|E(\mathbb{F}_p)|}{p}$$
  & conjecture predicts
  $$\prod_{p \le X} \frac{|E(\mathbb{F}_p)|}{p} = C_E \cdot (\log X)^r$$

- $$L^*(E,1) = \Omega_E \cdot \frac{|Ш(E)|}{|E(\mathbb{Q})^{\text{tors}}|^2} \cdot c_\infty(E) \cdot \prod_{p \mid \Delta} c_p(E).$$

What about $L(E,2)$?

Thm (Gross, 1979). $E = EC/\mathbb{Q}$ with CM

$L(E,2)$ explicitly related to



Explicit example: $E: x^3 + y^3 = 1$ w/ CM by $\mathbb{Q}(\zeta_3)$

Thm (Otsubo, 2011). Can take $f_1 = 1-x$, $f_2 = 1-y$.

More generally: $x^d + y^d = 1$ Fermat curve.

Ongoing work with A'Campo: $x_1^d + \cdots + x_n^d = 1$ Fermat hypersurface.